PHYC 511
Spring 2019

Lecture 7 02/06/2019

Review of Electrostatics (Cont'd)

Electrostatics in Dielectrics

When an electric field is applied to a medium, the atomic/molecular constituents have bound charges that get displaced and their centers separate. This is the "polarization" process, which results in formation of electric dipole moments throughout the medium.

Polarization results in volume and surface densities for the bound charge, ρ_{bound} and σ_{bound} respectively, according to:

$\vec{\nabla} \cdot \vec{P} = -\rho_{bound}$, $\vec{P} \cdot \hat{n} = \sigma_{bound}$

Here $\vec{P}$ is the electric dipole moment and $\hat{n}$ is the unit vector normal to the surface of the dielectric. Since $\vec{E}$ is related to the total charge density $\rho_{tot} = \rho + \rho_{bound}$, we have:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} (\rho - \vec{\nabla} \cdot \vec{P}) \Rightarrow \vec{\nabla} \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho$$

For linear response of the medium, we have:

$\vec{P} = \varepsilon_0 \chi_e \vec{E}$ (χ_e: electric susceptibility)

Then:

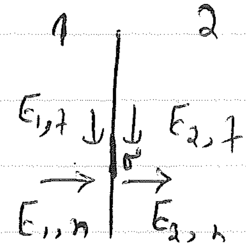
$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1+\chi_e) \vec{E} = \varepsilon \vec{E} \Rightarrow \vec{\nabla} \cdot \vec{D} = \rho, \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

Here $\varepsilon \equiv \varepsilon_0 (1+\chi_e)$ is electric permittivity of the medium, and

$\frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$ is the dielectric constant of the medium.

The boundary conditions at the interface of dielectric media follow from $\vec{\nabla} \cdot \vec{D} = \rho$ and $\vec{\nabla} \times \vec{E} = 0$:

$$\begin{cases} D_{2,n} - D_{1,n} = \sigma \\ E_{2,t} - E_{1,t} = 0 \end{cases} \Rightarrow \begin{cases} \varepsilon_2 E_{2,n} - \varepsilon_1 E_{1,n} = \sigma \\ E_{2,t} = E_{1,t} \end{cases}$$



Note that $\varepsilon = \varepsilon_0$ corresponds to the vacuum and $\varepsilon \to \infty$ describes a conductor.

The electrostatic energy in a dielectric medium follows the general expression $U = \frac{1}{2} \int_V \rho \Phi \, d\tau$. After using $\vec{\nabla} \cdot \vec{D} = \rho$ and

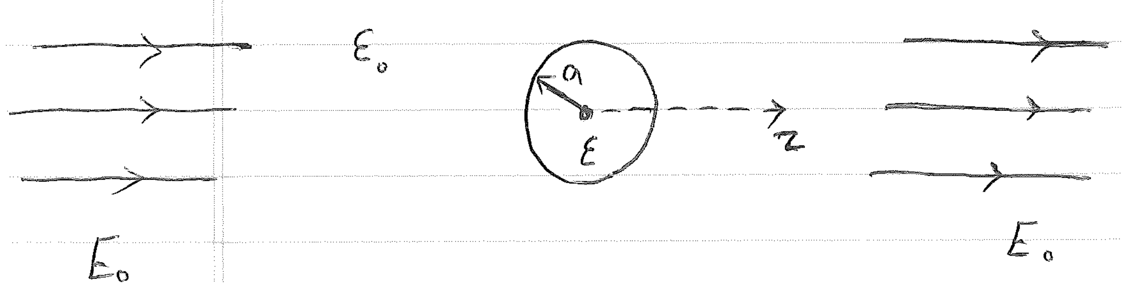
$\vec{E} = -\vec{\nabla}\Phi$, it can be shown that:

$$U = \int_V \frac{1}{2}(\vec{D}\cdot\vec{E})\,d\tau$$

Thus the electrostatic energy density is given by $\frac{1}{2}\vec{D}\cdot\vec{E}$ (which is $\frac{1}{2}\varepsilon|\vec{E}|^2 = \frac{|\vec{D}|^2}{2\varepsilon}$) inside a dielectric.

We now consider examples of boundary value problems involving dielectric bodies. An important difference from problems with conducting bodies is that the boundary is not an equipotential surface in this case. Insted, one has to solve the outside and inside problems and use the conditions $E_{in,t} = E_{out,t}$ and $\varepsilon_{out}E_{out,n} - \varepsilon_{in}E_{in,n} = \sigma$ to match the solutions at the boundary.

Example: Dielectric sphere in a uniform external electric field.



Inside the sphere, $\vec{D} = \epsilon\vec{E}$ and $\rho = 0$. Hence, $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon}\vec{\nabla} \cdot \vec{D} = 0$,

and Φ_{in} follows:

$$\Phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Here, we have used the fact that Φ_{in} must be finite at the origin. Also, with our choice of z axis, there is no ϕ dependence, which implies that only terms with $m=0$ are present in the expression (note that $Y_{l0}(\theta,\phi) = P_l(\cos\theta)$).

Outside the sphere, $\rho_{tot} = 0$, and therefore:

$$\Phi_{out} = \sum_{l=0}^{\infty} [B_l r^l + C_l r^{-(l+1)}] P_l(\cos\theta)$$

The asymtotic behavior of Φ_{out} at $r \to \infty$ must be that of a uniform field E_0. Hence, $B_l = 0$ for $l \geq 2$ and $B_1 = -E_0$.

The individual coefficients that are left will be determined by using the following relations:

(since there is no free charge)

$$E_{in,t} = E_{out,t} \quad , \quad \epsilon_0 E_{out,n} \overset{\downarrow}{=} \epsilon E_{in,n}$$

We note that at $r=a$ we have:

$$E_t = E_\theta = \frac{-1}{a}\frac{\partial \Phi}{\partial \theta}\Big|_{r=a} \quad , \quad E_n = E_r = \frac{-\partial \Phi}{\partial r}\Big|_{r=a}$$

After using the expressions for Φ_{in} and Φ_{out} on the previous page, these relations result in:

$$A_1 = \left(\frac{-3}{2+\frac{\varepsilon}{\varepsilon_0}}\right) E_0 \; , \quad C_1 = \left(\frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}\right) a^3 E_0 \quad , \quad A_\ell = C_\ell = 0 \quad \ell \geq 2$$

Therefore:

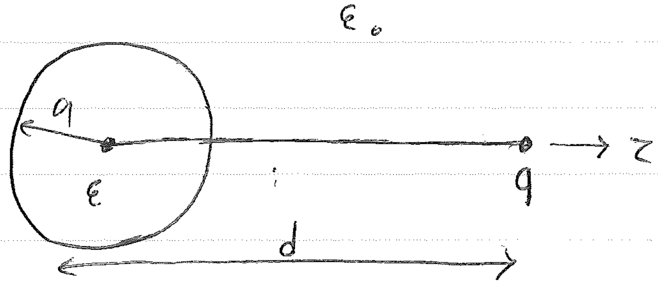
$$\boxed{\Phi_{in} = \left(\frac{-3}{2+\frac{\varepsilon}{\varepsilon_0}}\right) E_0 r \cos\theta \quad , \quad \Phi_{out} = -E_0 r\cos\theta + \left(\frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}\right) E_0 \frac{a^3}{r^2}\cos\theta}$$

We thus have a uniform electric field of strength $\frac{3}{2+\frac{\varepsilon}{\varepsilon_0}} E_0$ inside the sphere, resulting in the following polarization:

$$\vec{P} = (\varepsilon - \varepsilon_0)\vec{E} = 3\varepsilon_0\left(\frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}\right) E_0 \hat{z}$$

Note that in the limit $\varepsilon \to \infty$, we have $\vec{E}_{in} = 0$, as expected for a conducting sphere. In this case, $\vec{E}_{out}$ matches the result that we obtained for a conducting sphere in a uniform electric field.

Example: Point charge q in front of a dielectric sphere.



The potential outside the sphere follows:

$$\Phi_{out} = \frac{q}{4\pi\varepsilon_0 |\vec{x}-\vec{x}_0|} + \sum_{l=0}^{\infty} B_l \, r^{-(l+1)} P_l(\cos\theta)$$

The first term is due to the charge q and the second term represents the effect of bound charges on the surface of the sphere. Note that, due to localized nature of the distribution, $\Phi_{out} \to 0$ as $r \to \infty$.

The potential inside the sphere is given by:

$$\Phi_{in} = \sum_{l=0}^{\infty} A_l \, r^l P_l(\cos\theta)$$

Here, only positive powers of r are present so that Φ_{in} remains finite at $r=0$. We note that Φ_{in} is a solution to the Laplace equation as $\rho_{tot}=0$ anywhere inside the sphere.

Again, we match the inside and outside solutions at $r=a$ according to:

$$-\frac{1}{a}\frac{\partial \Phi_{in}}{\partial \theta}\Big|_{r=a} = -\frac{1}{a}\frac{\partial \Phi_{out}}{\partial \theta}\Big|_{r=a} \quad , \quad -\varepsilon \frac{\partial \Phi_{in}}{\partial r}\Big|_{r=a} = -\varepsilon_0 \frac{\partial \Phi_{out}}{\partial r}\Big|_{r=a}$$

At $r=a$, we have:

$$\frac{1}{|\vec{x}-\vec{x}_0|} = \sum_{l=0}^{\infty} \frac{r^l}{d^{l+1}} P_l(\cos\theta)$$

The first condition above (coming from $E_{in,t} = E_{out,t}$) gives:

$$A_l a^l = B_l a^{-(l+1)} + \frac{q a^l}{4\pi\varepsilon_0 d^{l+1}}$$

The second condition results in:

$$\varepsilon l A_l a^{l-1} = -\varepsilon_0 B_l (l+1) a^{-(l+2)} + \frac{q}{4\pi} \frac{l a^{l-1}}{d^{l+1}}$$

Then:

$$\boxed{A_l = \frac{q l}{4\pi[\varepsilon l + \varepsilon_0 (l+1)] d^{l+1}} \quad , \quad B_l = -\frac{q a^{2l+1} (\varepsilon_0 + \varepsilon l)}{4\pi\varepsilon_0 [\varepsilon l + \varepsilon_0 (l+1)] d^{l+1}}}$$

Substituting these in the expressions for Φ_{in} and Φ_{out} allows us to determine the potential everywhere in the space.